Tutorial Note V

The solutions of exercise 5 and exercise 6 are in the solutions of homework I. We only present the proof of Hopf lemma in this notes, as a complement to exercise 5, because the key idea of exercise 5 that $(a, b) \cdot \nabla u \geq 0$ at the maximum point is related to the Hopf lemma.

Theorem 0.1 (Hopf Lemma)
Suppose that $u \in C^2(\overline{B_R})$ is harmonic in B_R, attains its maximum at $x_0 \in \partial B_R$, and is not a constant. Then $\frac{\partial u}{\partial n}(x_0) > 0$.

Remark 0.1
Note that it is easy to see that $\frac{\partial u}{\partial n} \geq 0$. The key of the Hopf lemma is that the normal derivative is strictly greater than 0.

Proof. Suppose that the maximum of u is M. By the strong maximum principle, $u < M$ in B_R. Consider

$$v = u + \varepsilon(\mathrm{e}^{-\alpha|x|^2} - \mathrm{e}^{-\alpha R^2}),$$

where ε is small and α is large and they will be determined later.

$$\Delta v = \varepsilon \mathrm{e}^{-\alpha|x|^2}(4\alpha^2|x|^2 - 2n\alpha).$$

On the annulus $A(R/2, R)$,

$$4\alpha^2|x|^2 - 2n\alpha \geq \alpha^2 R^2 - 2n\alpha.$$

So if we take α large enough,

$$\Delta v \geq 0$$

on $A(R/2, R)$. On $\partial B_{R/2}$,

$$v = u + \varepsilon(\mathrm{e}^{-\alpha(R/2)^2} - \mathrm{e}^{-\alpha R^2}).$$

Since $u < M$ on $\partial B_{R/2}$,

$$\max_{\partial B_{R/2}} u < M.$$

So we could take ε small enough such that

$$\max_{\partial B_{R/2}} u + \varepsilon < M.$$

Now apply weak maximum principle to v on $A(R/2, R)$. Since $v = u$ on ∂B_R and $v < M$ on $\partial B_{R/2}$, $v \leq M$ on $A(R/2, R)$. So v attain its maximum at x_0, and

$$\frac{\partial v}{\partial n}(x_0) \geq 0.$$

So

$$\frac{\partial v}{\partial n}(x_0) = \frac{\partial u}{\partial n}(x_0) - 2\varepsilon\alpha \mathrm{e}^{-\alpha R^2} R \geq 0.$$

Therefore,

$$\frac{\partial u}{\partial n}(x_0) > 0.$$

□